Ising model on two connected Barabasi-Albert networks

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We investigate analytically the behavior of the Ising model on two connected Barabasi-Albert networks. Depending on relative ordering of both networks there are two possible phases corresponding to parallel or antiparallel alignment of spins in both networks. A difference between critical temperatures of both phases disappears in the limit of vanishing inter-network coupling for identical networks. The analytic predictions are confirmed by numerical simulations.

I. INTRODUCTION

The Ising model on Barabasi-Albert (BA) scale-free network [1] has been investigated both numerically [2] as well as analytically [3] and it has been shown that the critical temperature of such a system is proportional to the logarithm of a total spins number N. Similar the Ising models have been investigated for a general class of random scale-free graphs [4–6] and it has been shown that critical temperatures of such systems depend substantially on a characteristic exponent γ describing the probability distribution of node degrees. Other studies of this model include investigations of antiferromagnetic interactions [7], dynamics on directed networks [8] and critical properties of spin glasses [9].

Besides a large interest in physical properties of Ising-like models, they also seem to be important for opinion formation modeling [10–13]. The Ising model exhibits a majority rule dynamics—the feature that often can be found in social systems, where a given person changes his/her opinion to fit to a majority of his neighbors. Since it is common that social networks have modular structure of weakly coupled clusters [14], it is of particular interest to see how the Ising model behaves in the case of two interacting complex networks. While geometrical properties of interconnected complex networks have been studied before [15], the dynamics in such systems has not been explored thoroughly.

We start with analytical investigations of the Ising model on two interconnected Barabasi-Albert networks, and then show results of numerical simulations confirming our analytical studies.

II. MODEL

Our model considers Ising spins on a BA network. Each node of the network has a spin. We study only ferromagnetic interactions existing between directly connected spins.

The BA model is a model of a growing network [1]. One starts with m fully connected nodes, and adds new nodes to the network. Each new node creates m connections to the existing network. The probability that a connection will be made to a node i is proportional to its degree k_i. This results in a scale-free network, with a degree distribution $P(k) \sim k^{-3}$. We assume that two BA networks are connected by E_{AB} links (Fig. 1). Each of these links connects a node in network A with a node in network B. We choose a node in network A preferentially, i.e., the probability to pick a given node i equals $\Pi_{Ai} = k_{AAi}/\Sigma_j k_{AAj}$. Note that we are using intra-network node degree k_{AAi} here, not the total node degree. We choose the node in network B in the same way and connect two chosen nodes by a link. This means that an inter-network degree k_{AB} of a node in the network A is proportional to an intra-network node degree k_{AA} on average. A similar relation holds for degrees in the network B.

III. ANALYTIC CALCULATIONS

The problem of the Ising model in a single BA network was already solved analytically by Bianconi by an appropriately tailored mean-field approach [3]. We use a similar approach for the problem of two connected networks.

The Hamiltonian of the Ising model for a single BA network can be written as

$$H = - \sum_{i,j} J_{ij} s_i s_j - \sum_i h_i s_i, \qquad (1)$$

where $s_i, s_j = \pm 1$ are spins of nodes i, j, a constant J_{ij} is a ferromagnetic coupling between them and h_i is an external field acting on spin i. The coupling constants J_{ij} are equal to a positive constant if the spins are connected, and are zero otherwise.
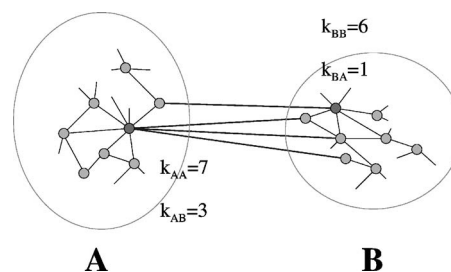

FIG. 1. Two connected BA networks. A few nodes from each network are shown. The intra-network degrees k_{AA} and k_{BB} as well as inter-network degrees k_{AB} and k_{BA} for two sample nodes are presented.

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The exact solution for the average spin $\langle s_i \rangle$ in a single network can be written as

$$\langle s_i \rangle = \left\langle \tanh\left(\beta \sum_j J_{ij} s_j + \beta h_i \right) \right\rangle, \quad (2)$$

where $\beta = 1/T$, the temperature T is measured in units of inverse Boltzmann constant $1/k_B$ and averaging is over the canonical ensemble. If we now consider an average over all possible realizations of BA networks, then $\langle J_{ij} \rangle = J k_i k_j / E$. We use the mean field approximation, taking $\langle J_{ij} \rangle$ in place of J_{ij} in our equation. Since i and j are ordered pairs, the total number of pairs E is twice the number of edges in the network. If we take the external field h_i equal to zero, Eq. (1) has the following form:

$$\langle s_i \rangle = \left\langle \tanh\left[\beta J \sum_j \left(\frac{k_i k_j}{E} s_j \right) \right] \right\rangle. \quad (3)$$

Now we consider a pair of coupled networks A and B. The parameters describing both networks can be split into four groups—two describe internal properties of each network, and two describe network-network interactions. We introduce the following notation: s_{Ai} and s_{Bi} are spins in networks A and B, J_{AA} and J_{BB} are coupling constants between spins in networks A and B, respectively, $J_{AB} = J_{BA}$ are the coupling constants between spins in different networks, k_{AAi} and k_{BBi} are intra-network node degrees, k_{ABi} and k_{BAi} are inter-network node degrees, E_{AA} and E_{BB} are twice the total numbers of all intra-network links in A and B, $E_{AB} = E_{BA}$ is the number of links between the networks.

Now we extend Eq. (3), introducing the influence of the second network. This way we obtain two equations for average spins in every network. We use a standard mean-field approach, and approximate the hyperbolic tangent by a linear function. Since average spins are relatively close or equal to zero near our critical points, the approximation does not introduce large errors. As result we get

$$\langle s_{Ai} \rangle = \beta J_{AA} k_{AAi} \sum_j \frac{k_{AAj} \langle s_{Aj} \rangle}{E_{AA}} + \beta J_{BA} k_{ABi} \sum_j \frac{k_{BAj} \langle s_{Bj} \rangle}{E_{BA}} \quad (4)$$

$$\langle s_{Bi} \rangle = \beta J_{BB} k_{BBi} \sum_j \frac{k_{BBj} \langle s_{Bj} \rangle}{E_{BB}} + \beta J_{AB} k_{BAi} \sum_j \frac{k_{ABj} \langle s_{Aj} \rangle}{E_{AB}}. \quad (5)$$

To get a relation for the system critical temperature we need to have a self-consistent equations for order parameter. The case of a single network required introduction of only single weighted spin $S = \sum_i k_i \langle s_i \rangle / E$, where the $\langle s_i \rangle$ is mean-field average for a given spin i. In the case of two connected networks, we need to consider four such weighted spins S_{AA}, S_{BB}, S_{AB} and S_{BA}

$$S_{AA} = \sum_i k_{AAi} \langle s_{Ai} \rangle / E_{AA}, \quad (6)$$

$$S_{BB} = \sum_i k_{BBi} \langle s_{Bi} \rangle / E_{BB}, \quad (7)$$

$$S_{AB} = \sum_i k_{ABi} \langle s_{Ai} \rangle / E_{AB}, \quad (8)$$

$$S_{BA} = \sum_i k_{BAi} \langle s_{Bi} \rangle / E_{BA}. \quad (9)$$

S_{AA} and S_{BB} hold the same meaning as for a single network, S_{AB} is the mean weighted spin of the network A observed by spins in the network B, while S_{BA} is a mean weighted spin of the network B observed by spins in the network A.

Equation (10)–(13) were received from Eqs. (4) and (5) by multiplying by appropriate factors [see Eqs. (6)–(9)] and summing over i. These four equations contain only four weighted spins as unknown collective variables and are approximate mean-field description of the system close to a critical point. It follows we receive

$$S_{AA} = \beta J_{AA} S_{AA} \sum_i \frac{k_{AAi}^2}{E_{AA}} + \beta J_{BA} S_{BA} \sum_i \frac{k_{ABi} k_{AAi}}{E_{AA}}, \quad (10)$$

$$S_{BB} = \beta J_{BB} S_{BB} \sum_i \frac{k_{BBi}^2}{E_{BB}} + \beta J_{AB} S_{AB} \sum_i \frac{k_{BAi} k_{BBi}}{E_{BB}}, \quad (11)$$

$$S_{AB} = \beta J_{AA} S_{AA} \sum_i \frac{k_{AAi} k_{ABi}}{E_{AB}} + \beta J_{BA} S_{BA} \sum_i \frac{k_{ABi}^2}{E_{AB}}, \quad (12)$$

$$S_{BA} = \beta J_{BB} S_{BB} \sum_i \frac{k_{BAi} k_{BBi}}{E_{BA}} + \beta J_{AB} S_{AB} \sum_i \frac{k_{BAi}^2}{E_{BA}}. \quad (13)$$

If we assume that $k_{ABi} = p_A k_{AAi}$ and $k_{BAi} = p_B k_{BBi}$, which means that the number of links outside the network is proportional to the number of links within the network, we can greatly simplify our four equations. This assumption is valid when one takes into account the way we create inter-network links in our model. The probabilities p_A and p_B are fixed numbers, although the values are not independent and are connected with the number of links between networks in following way: $p_A E_{AA} = E_{AB} = E_{BA} = p_B E_{BB}$.

Using this assumption, we do not need to consider the cross-network weighted spins S_{AB} and S_{BA} as they are proportional to S_{AA} and S_{BB}. Now our first two equations become

$$S_A = \beta J_{AA} S_A \sum_i \frac{k_{AAi}^2}{E_{AA}} + \beta J_{BA} S_B \sum_i \frac{k_{AAi}^2 p_A}{E_{AA}} \quad (14)$$

$$S_B = \beta J_{BB} S_B \sum_i \frac{k_{BBi}^2}{E_{BB}} + \beta J_{AB} S_A \sum_i \frac{k_{BBi}^2 p_B}{E_{BB}}, \quad (15)$$

where $S_A \equiv S_{AA}$ and $S_B \equiv S_{BB}$. The equation array can be written as a single matrix equation

$$\mathbf{S} = \beta \hat{\Lambda} \mathbf{S}, \quad (16)$$

where $\mathbf{S}$ is a vector $\begin{bmatrix} S_A \\ S_B \end{bmatrix}$ describing the state of the system and $\hat{\Lambda}$ is a matrix describing effective interaction strengths between spins belonging to the same or to different networks

$$\hat{\Lambda} = \begin{bmatrix} \Lambda_{AA} & \Lambda_{BA} \\ \Lambda_{AB} & \Lambda_{BB} \end{bmatrix} = \begin{bmatrix} J_{AA}\dfrac{\langle k_{AA}^2 \rangle}{\langle k_{AA} \rangle} & p_B J_{BA}\dfrac{\langle k_{AA}^2 \rangle}{\langle k_{AA} \rangle} \\ p_A J_{AB}\dfrac{\langle k_{BB}^2 \rangle}{\langle k_{BB} \rangle} & J_{BB}\dfrac{\langle k_{BB}^2 \rangle}{\langle k_{BB} \rangle} \end{bmatrix}. \quad (17)$$

In the case of a single network A, solutions other than $S_A=0$ can exist only if $\beta J_{AA}\frac{\langle k_{AA}^2 \rangle}{\langle k_{AA} \rangle} > 1$ [3]. In the case of two coupled networks, this condition corresponds to an eigenvalue of Eq. (16) greater than 1. The eigenvalues are

$$\lambda_\pm = \frac{\beta}{2}(\Lambda_{AA} + \Lambda_{BB} \pm \sqrt{(\Lambda_{AA} - \Lambda_{BB})^2 + 4\Lambda_{BA}\Lambda_{AB}}). \quad (18)$$

Comparing these eigenvalues with 1, we get the following critical temperatures:

$$T_{c\pm} = \frac{\Lambda_{AA} + \Lambda_{BB} \pm \sqrt{(\Lambda_{AA} - \Lambda_{BB})^2 + 4\Lambda_{BA}\Lambda_{AB}}}{2}. \quad (19)$$

Since the diagonal elements of $\hat{\Lambda}$, Λ_{AA}, and Λ_{BB} are critical temperatures T_{cA}, T_{cB} for separate networks we can write the critical temperatures for the coupled system as

$$T_{c\pm} = \frac{T_{cA} + T_{cB} \pm \sqrt{(T_{cA} - T_{cB})^2 + 4\Lambda_{BA}\Lambda_{AB}}}{2}. \quad (20)$$

To better understand the meaning of these solutions, we introduce the following variables:

$$\mathcal{A} = (T_{cA} + T_{cB})/2 \quad (21)$$

$$\mathcal{D} = (T_{cA} - T_{cB})/2 \quad (22)$$

$$\mathcal{C} = \sqrt{\Lambda_{BA}\Lambda_{AB}}. \quad (23)$$

The value $\mathcal{A}$ ("average") describes average critical temperatures of the networks, $\mathcal{D}$ ("difference") is the difference between critical temperatures of both networks, $\mathcal{C}$ ("coupling") describes a strength of inter-network interactions.

Using this notation the critical temperatures can be written shortly as

$$T_{c\pm} = \mathcal{A} \pm \sqrt{\mathcal{D}^2 + \mathcal{C}^2}. \quad (24)$$

Let us now consider eigenvectors associated with $\lambda_\pm$. They are proportional to the magnetization of both networks that appears below a given critical temperature and disappears above it. The un-normalized eigenvectors are

$$\mathbf{S}_\pm = \begin{bmatrix} 1 \\ \dfrac{-\mathcal{D} \pm \sqrt{\mathcal{D}^2 + \mathcal{C}^2}}{\Lambda_{BA}} \end{bmatrix}. \quad (25)$$

The eigenvector $\mathbf{S}_-$ has opposite signs of its components and corresponds to networks ordered with antiparallel weighted spins, while the eigenvector $\mathbf{S}_+$ has the same signs of the components and corresponds to networks ordered with parallel weighted spins.

In the limit of vanishing inter-network coupling ($\mathcal{C}=0$) the eigenvalues are simply the diagonal elements of the matrix
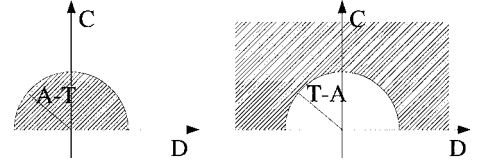

FIG. 2. The area of parameters $\mathcal{C}$ and $\mathcal{D}$ where the ordered states $\mathbf{S}_\pm$ can exist. The order appears in the dashed areas. The state $\mathbf{S}_-$ of antiparallel ordering corresponds to the left picture, the state $\mathbf{S}_+$ or parallel ordering corresponds to the right picture. The radius of the circle is the difference $T-\mathcal{A}$. Note that solution $\mathbf{S}_-$ does not exist for $\mathcal{A}<T$ and solution $\mathbf{S}_+$ exist for any $\mathcal{C}$ and $\mathcal{D}$ for $\mathcal{A}>T$. In the case $\mathcal{C}=0$ the networks do not interact at all, and the solutions lose their normal meaning.

$\lambda_A=T_{cA}$, $\lambda_B=T_{cB}$ and the associated normalized eigenvectors are $\mathbf{S}_A=\begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{S}_B=\begin{bmatrix}0\\1\end{bmatrix}$. This means that in this limit two stable states of the system correspond to the ordering of just one of the networks, and there is no relation between the order in each network. It shows our approach gives correct results in this specific case. Now let us investigate the inequality conditions for existence of solutions $\mathbf{S}_\pm$. From the condition $\lambda_\pm > 1$ we have

$$\mathcal{A} - T > \sqrt{\mathcal{D}^2 + \mathcal{C}^2}, \quad (26)$$

$$T - \mathcal{A} < \sqrt{\mathcal{D}^2 + \mathcal{C}^2}. \quad (27)$$

The meaning of these inequalities is presented at Fig. 2.

If we consider networks of the same size, the dependence of critical temperatures on inter-network interaction strength $\mathcal{C}$ is linear, and both critical temperatures are the same for $\mathcal{C} \rightarrow 0$. In such case the system critical temperature $\mathcal{T}_{c\pm} = \mathcal{A} = T_{cA} = T_{cB}$ (Fig. 3).

The analytic results hold true for any random network, where the probability of a link existing between any two nodes i and j is proportional to the product $k_i k_j$. This is the only assumption about network structure we have used, so
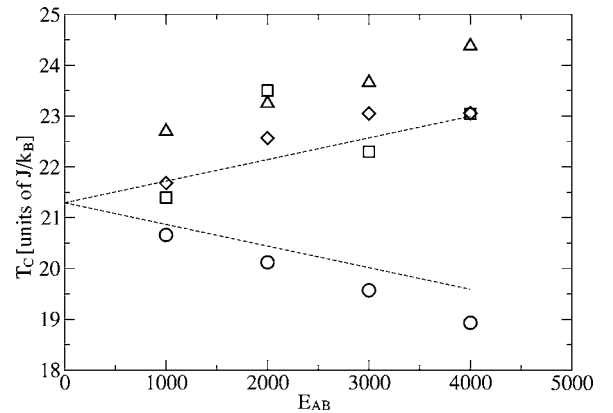

FIG. 3. The dependence of temperatures $T_{c\pm}$ on number of links between networks E_{AB}. The dashed lines are analytic predictions [Eq. (19)]. The symbols are numeric results. Circles correspond to T_{c-} [J/k_B]. Squares and diamonds correspond to T_{c+} [J/k_B] and are calculated from susceptibility $\chi=(S_h-S_0)/h$. Triangles correspond to T_{c+} [J/k_B] and are calculated from susceptibility $\chi \sim \langle S^2 \rangle - \langle S \rangle^2$.

any networks where the condition is fulfilled (i.e., random networks) are subject to our analytical predictions.

Our numerical calculations, found in the following section, correspond to the specific case of BA network and constant coupling $J_{AA}=J_{AB}=J_{BA}=J_{BB}=J$. We can write the critical temperatures as follows:

$$T_{c\pm} = \frac{T_{cA} + T_{cB}}{2} \pm \sqrt{\left(\frac{T_{cA} - T_{cB}}{2}\right)^2 + p_A p_B T_{cA} T_{cB}}, \quad (28)$$

where $T_{cA}=J(m_A/2)\ln N_A$ and $T_{cB}=J(m_B/2)\ln N_B$. N_A is the number of nodes in network A, m_A is the number of links each new node creates (see BA network creation in Sec. III). Similar for network B.

IV. NUMERIC RESULTS

Our analytic calculations show the existence of two different ordered states and estimate values of two critical temperatures where these states disappear. Below we investigate numerically a case of two coupled BA networks with the same number of nodes $N_A=N_B=5000$ and links $E_{AA}=E_{BB}=50\,000$ ($\langle k_{AA}\rangle=\langle k_{BB}\rangle=10$).

We run Ising dynamics on these networks, setting the following initial condition: all spins in both networks have the same value $s_{Ai}=s_{Bi}=+1$. We allow the system to relax for $\tau=20$ time steps, perform averaging for $\tau=20$ time steps, then increase the temperature and start from the same initial condition as before. This way, results for different temperatures are not correlated. We find the weighted spin $S=S_A+S_B$ for each temperature T and average it over 100 network realizations. Simulations are performed for different numbers of inter-network links $E_{AB}=1000, 2000, 3000$ and 4000 that were attached preferentially to fulfill the assumptions of our model (see the discussion at the end of Sec. II).

For low temperatures T the system is ordered. As the temperature increases the average weighted spin S decreases. When the temperatures increase over T_{c+}, the ferromagnetic ordered state of both networks disappears, i.e., both networks become paramagnetic (Fig. 4).

Finding the exact value of critical temperature T_{c+} from numerical simulations is not straightforward. If one observes the dependence of the weighted magnetization on rising temperature and tries to fit the magnetization decay to a linear or to an exponential function, the results strongly depend on relaxation time τ. To overcome this problem we observed the temperature dependence of the system *susceptibility χ*. In fact, by comparison to standard models of magnetic systems, one can expect that the initial susceptibility diverges at $T=T_{c+}$. In our finite system we are looking simply for the maximum of χ. To estimate χ we are using two methods. First we compare average weighted spin S for a small external field $h=0.05J$ and the value of S_0 with no external field. It follows $\chi=(S_h-S_0)/h$. Because such results are strongly fluctuating as a function of system history and temperature (Fig. 4), we calculate running average over 30 temperature points and find the maximum of χ by fitting a parabolic curve. The top of the parabola corresponds to the position of the critical temperature T_{c+}. We found that these values are
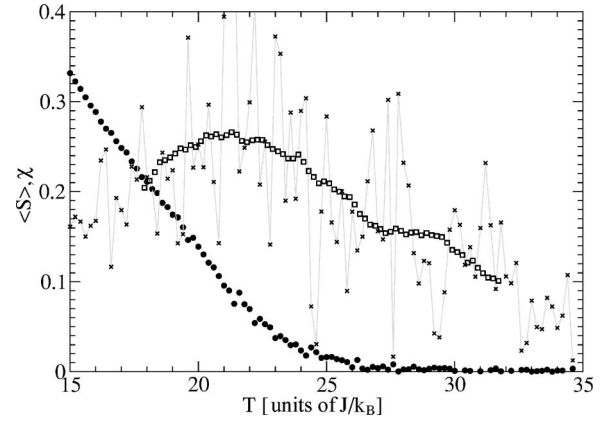

FIG. 4. The dependence of the average weighted spin $S=S_A+S_B$ and its susceptibility for small external field $h=0.05J$ in the case of two BA networks with $E_{AB}=1000$ connections between them. An initial condition for each temperature is a completely ordered system. The full symbols depict S, the X symbols correspond to susceptibility $\chi=(S_{0.05}-S_0)/h$ (the lines are just to guide eye), the empty symbols are 30-point running average of the susceptibility. The parabolic fit was used to find the susceptibility maximum.

independent of the relaxation time τ used in our numerical experiment. The second method of finding the critical temperature T_{c+} is observation of the time average $\langle S^2\rangle-\langle S\rangle^2$, where we average over one relaxation period τ. The magnitude of the fluctuations is proportional to the susceptibility $\chi \sim \langle S^2\rangle-\langle S\rangle^2$ according to the fluctuation-dissipation theorem [16]. Similarly to the previous method, we calculate running average over ten points and find the maximum. The values are shifted by a constant value comparing to analytic results and do not fluctuate as much as those obtained from the first method (see Fig. 3).

Now we consider the same networks with the following initial condition for each temperature: spins in both networks
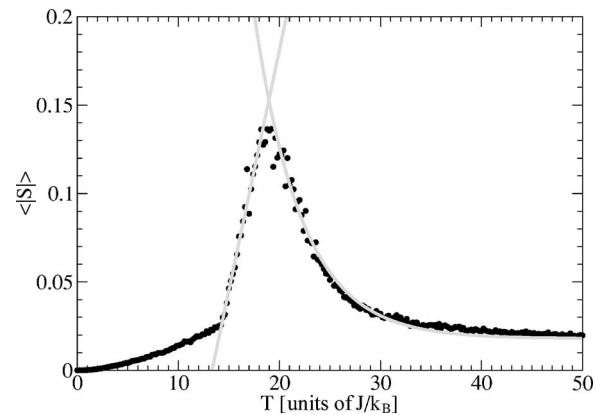

FIG. 5. The temperature dependence of the average weighted spin value $|S|=|S_A+S_B|$ in the case of two BA networks with $E_{AB}=4000$ connections between them. Initial conditions for each temperature are two fully ordered networks with opposite spins. The symbols correspond to numerical data and gray lines are approximations of magnetization behavior below and above critical temperature.
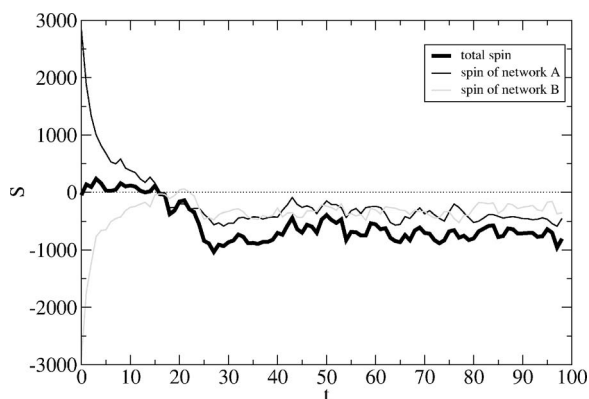
FIG. 6. An example of time evolution of magnetization for two networks of size $N_A=N_B=5000$ and $\langle k \rangle=4$ with $E_{AB}=500$ links between them in temperature $T=30$ (scaled in k_B/J). At this temperature, the antiparallel ordered state is not stable, and the initial condition of networks ordered in such fashion evolves towards a stable parallel ordered state.

are ordered antiparallel $s_{Ai}=+1, s_{Bi}=-1$. The relaxation algorithm and the measurement procedure are the same as for parallel spin case, however now we consider absolute values of weighted spin of the whole system $|S|=|S_A+S_B|$. For low temperatures T both networks remain ordered in opposite directions (Fig. 5) and as result an average weighted spin value $|S|=|S_A+S_B|$ fluctuates around 0. When the temperature increases over T_{c-}, the state of antiparallel ordered networks disappears and networks start to order in a parallel fashion. An example of such a scenario is presented at Fig. 6. When temperature further increases and $T>T_{c+}$ the system becomes paramagnetic.

We found numerically the critical temperature T_{c-} from the intersection of extrapolations of rising and declining part of the curve $S(T)$ (linear fit was used for the rising part and exponential fit for the declining part) (Fig. 5). This point slightly depends on relaxation times τ. For large τ the effect of finite system size can be easily observed and the system jumps from an antiparallel state to a parallel one that has a lower energy.

The results for dependence of T_{c-} and T_{c+} on the number of links between networks E_{AB} agree with the analytic calculations (Fig. 3).

V. CONCLUSIONS

In the system of two BA networks, the Ising model possesses two low-temperature stable states — both networks ordered parallel or antiparallel. It follows there are two critical temperatures corresponding to the disappearance of these two stable states—T_{c-} and T_{c+}. They are placed symmetrically around the average of critical temperatures of separate networks. The difference between them depends on density of inter-network links and the difference between critical temperatures of separate networks. The analytic calculations agree with performed numeric simulations.

ACKNOWLEDGMENTS

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